Math 210C Lecture 16 Notes

Daniel Raban

May 6, 2019

1 Injective Modules and Exact Functors

1.1 Baer's criterion and divisible groups

Theorem 1.1. *Let R be a PID. Then every projective R-module is free.*

Theorem 1.2. *Let R be a Dedekind domain. Then every ideal of R is projective.*

Recall that an object I is injective if for every monomorphism $\iota : A \to V$ and $f : A \to I$, there exists a $g : B \to I$ such that $g \circ \iota = f$:

$$\begin{tikzcd} & & I \\ 0 \arrow[r] & A \arrow[ur, "f"] \arrow[r, "\iota"'] & B \arrow[u, dashed, "g"'] \end{tikzcd}$$

Example 1.1. $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.

Example 1.2. $\mathbb{Z}$ is not an injective $\mathbb{Z}$-module: $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$ is not split.

Proposition 1.1 (Baer's criterion)**.** *A left R-module I is injective if and only if fir every left ideal J of R, $J \to I$ can be extended to a morphism $R \to I$.*

$$\begin{tikzcd} & & I \\ 0 \arrow[r] & J \arrow[ur] \arrow[r] & R \arrow[u, dashed] \end{tikzcd}$$

Proof. ($\Longleftarrow$): Suppose $A \subseteq B$ are R-modules, and let $f : A \to I$. Let $X = \{(C, h) \mid h : C \to I, C \subseteq B, h|_A = f\}$. Establish the ordering $\leq$ on X by $(X, h) \leq (C', h')$ if $C \subseteq C'$ and $h'|_C = h$. Now if $\mathcal{C} \subseteq X$ is a chain, let $D = U_{(C,h)\in\mathcal{C}} C$. Let $k : D \to I$ be $k(d) = h(d)$ if $(c, h) \in \mathcal{C}$ and $d \in C$. Then (D, k) is an upper bound for $\mathcal{C}$. So X has a maximal element (M, ℓ) by Zorn's lemma. If $M \neq B$, let $b \in B \setminus M$. Let $J = \{r \in R : rb \in M\} \subseteq R$ be a left ideal. Let $s : J \to I$ be $s(r) = \ell(rb)$ for $r \in J$. This is an R-module homomorphism.

By assumption, there is a $t : R \to I$ extending s. Let $N = M + Rb \supsetneq M$; $N \subseteq B$. Define $q : N \to I$ by $q(m) = \ell(m)$ for $m \in M$ and $q(rb) = t(r)$ for $r \in R$. $M \cap Rb - Jb$, and if $r \in J$, then $\ell(rb) = s(r) = t(r)$; so q is well-defined. Then q is an R-module homomorphism extending ℓ to N, which is a contradiction. Therefore, $M = n$, so ℓ is the desired extension. □

Example 1.3. We can now show that $\mathbb{Q}$ is an injective $\mathbb{Z}$-module. We can extend $f : n\mathbb{Z} \to \mathbb{Q}$ to $g : \mathbb{Z} \to \mathbb{Q}$ by $g(1) = (1/n)g(n)$ for $n \neq 0$ and $g(1) = 0$ otherwise.

Example 1.4. $\mathbb{Z}/n\mathbb{Z}$ is an injective $\mathbb{Z}/n\mathbb{Z}$-module.

Example 1.5. $\mathbb{Z}/3\mathbb{Z}$ is not an injective $\mathbb{Z}/9\mathbb{Z}$-module. We cannot extend the isomorphism $3\mathbb{Z}/9\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}$ to all of $\mathbb{Z}/9\mathbb{Z}$

Definition 1.1. An abelian group D is **divisible** if for all $n \geq 1$, $n : D \to D$ sending $x \mapsto nx$ is surjective.

Proposition 1.2. *The injective objects in* Ab $= \mathbb{Z}$-Mod *are the divisible groups.*

Proof. Suppose D is injective. Let $d \in D$ and $n \geq 1$. We have $\phi : \mathbb{Z} \to D$ with $\phi(1) = D$.

$$\begin{tikzcd} & D \\ \mathbb{Z} \arrow[ur, "\phi"] \arrow[r, "n"'] & \mathbb{Z} \arrow[u, dashed] \end{tikzcd}$$

Since D is inejctive, there is a $\theta : \mathbb{Z} \to D$ such that $\theta(na) = \phi(a)$ or all $z \in \mathbb{Z}$. Then $n\theta(1) = \theta(n) = \phi(1) = d$. So D is divisible.

Suppose that D is divisible. Send $f : n\mathbb{Z} \to D$ be $n \mapsto d$. Then we can define $g : \mathbb{Z} \to D$ by $g(1) = e$ where $ne = d$ by the divisibility of D.

$$\begin{tikzcd} & D \\ n\mathbb{Z} \arrow[ur, "f"] \arrow[r] & \mathbb{Z} \arrow[u, dashed, "g"'] \end{tikzcd}$$

Then $g(b) = nbe = bd = f(b)$, so g extends f. By Baer's criterion, D is injective. □

1.2 Exact functors

Definition 1.2. Let $F : \mathcal{C} \to \mathcal{D}$ be an additive functor of abelian categories.

1. We say F is **left exact** if it preserves left short exact sequences: if

$$0 \longrightarrow A \longrightarrow B \longrightarrow C$$

is exact, then so is

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C)$$

2. We say F is **right exact** if it preserves right short exact sequences: if

$$A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact, then so is

$$F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0$$

3. We say F is **exact** if it preserves short exact sequences: if

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact, then so is

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0$$

Remark 1.1. A contravariant functor $F : \mathcal{C} \to \mathcal{C}$ is left exact if it turns right exact sequences into left exact sequences.

Example 1.6. The forgetful functor R-Mod $\to$ Ab is exact.

Example 1.7. if G is a group, let $\mathbb{Z}[G]$ be th group ring. Let A be s $\mathbb{Z}[G]$-module, and let $A^G = \{a \in A : ga = a \,\forall g \in G\}$. Let $F : \mathbb{Z}[G]\text{-Mod} \to \mathbb{Z}[G]\text{-Mod}$ send $A \mapsto A^G$. This is a left exact functor. However, it is not right exact.

Lemma 1.1. *Let $F : \mathcal{C} \to \mathcal{D}$ be an additive functor. Then F is left/right exact if and only if it sends short exact sequences to left/right short exact sequences.*

Lemma 1.2. *Let $F : \mathcal{C} \to \mathcal{D}$ be an additive functor. The following are equivalent:*

1. *F is exact.*
2. *F is left exact and right exact.*
3. *F preserves all 3-term exact sequences*

$$A \longrightarrow B \longrightarrow C$$

4. *F preserves all exact sequences.*

Proof. (3) $\implies$ (1): The sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact iff the following three sequences are exact:

$$0 \longrightarrow A \longrightarrow B$$

$$A \longrightarrow B \longrightarrow C$$

$$B \longrightarrow C \longrightarrow 0$$

So the following sequences are exact:

$$0 \longrightarrow F(A) \longrightarrow F(B)$$

$$F(A) \longrightarrow F(B) \longrightarrow F(C)$$

$$F(B) \longrightarrow F(C) \longrightarrow 0$$

We then get the following exact sequence:

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0$$

(1) $\implies$ (4): Suppose we have the exact sequence

$$\cdots \longrightarrow A_{i+1} \xrightarrow{d_{i+1}} A_i \xrightarrow{d_i} A_{i-1} \longrightarrow \cdots$$

Then we get the exact sequences

$$0 \longrightarrow \mathrm{im}(d_{i+1}) \longrightarrow A_i \longrightarrow \mathrm{im}(d_i) \longrightarrow 0$$

for each i. So

$$0 \longrightarrow F(\mathrm{im}(d_{i+1})) \longrightarrow F(A_i) \longrightarrow F(\mathrm{im}(d_i)) \longrightarrow 0$$

and we get that the sequence

$$\cdots \longrightarrow F(A_{i+1}) \xrightarrow{F(d_{i+1})} F(A_i) \xrightarrow{F(d_i)} F(A_{i-1}) \longrightarrow \cdots$$

is exact. $\square$

Lemma 1.3. *Let $X \in \mathcal{C}$. Then*

1. *The functor $h_X : \mathcal{C} \to \mathrm{Ab}$ sending $h_X(B) = \mathrm{Hom}_{\mathcal{C}}(X, B)$ is left exact.*
2. *The contravariant functor $h^X : \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ sending $h^X(A) = \mathrm{Hom}_{\mathcal{C}}(A, X)$ is left exact.*